TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 180, June 1973

ON SYMMETRIC ORDERS AND SEPARABLE ALGEBRAS

BY

T. V. FOSSUM[1]

ABSTRACT. Let K be an algebraic number field, and let Λ be an R-order in a separable K-algebra A, where R is a Dedekind domain with quotient field K; let Δ denote the center of Λ. A left Λ-lattice is a finitely generated left Λ-module which is torsion free as an R-module. For left Λ-modules M and N, $\mathrm{Ext}^1_\Lambda(M, N)$ is a module over Δ. In this paper we examine ideals of Δ which are the annihilators of $\mathrm{Ext}^1_\Lambda(M, \_)$ for certain classes of left Λ-lattices M related to the central idempotents of A, and we compute these ideals explicitly if Λ is a symmetric R-algebra. For a group algebra, these ideals determine the defect of a block. We then compare these annihilator ideals with another set of ideals of Δ which are closely related to the homological different of Λ, and which in a sense measure deviation from separability. Finally we show that, for Λ to be separable over R, it is necessary and sufficient that Λ is a symmetric R-algebra, Δ is separable over R, and the center of each localization of Λ at the maximal ideals of R maps onto the center of its residue class algebra.

1. **Annihilators of** Ext. We will adhere to the notation and assumptions in the introduction throughout this paper: That is, Λ is an R-order in a separable K-algebra A. If M and N are left Λ-modules, then $\mathrm{Ext}^1_\Lambda(M, N)$ is an $(\mathrm{End}_\Lambda(M), \mathrm{End}_\Lambda(N))$-bimodule. The ring homomorphism from Δ into $\mathrm{End}_\Lambda(M)$ given by left multiplication makes $\mathrm{Ext}^1_\Lambda(M, N)$ into a left Δ-module.

When M is a left Λ-lattice, we may identify M with a subset of $K \otimes_R M$ by the map $m \longrightarrow 1 \otimes m$, and under this identification we have that $KM = K \otimes_R M$. It is clear that KM is a left A-module, and in this context we may speak of aM for any $a \in A$. It is equally obvious that to each Λ-homomorphism $f: M \longrightarrow N$ of left Λ-lattices, there is a unique extension $f: KM \longrightarrow KN$ (which we still denote by f) which is an A-homomorphism. We will use these conventions frequently and without further mention.

Definition. For M a left Λ-lattice, set

$$\mathrm{Ann}_\Delta(\mathrm{Ext}^1_\Lambda(M,\_)) = \bigcap_N \{\mathrm{Ann}_\Delta(\mathrm{Ext}^1_\Lambda(M, N))\},$$

where the intersection is over all left Λ-modules N, and where $\mathrm{Ann}_\Delta(E) = \{z \in \Delta: zE = 0\}$ for any left Δ-module E. For a central idempotent e of A, define $J_\Delta(\Lambda, e) = \bigcap_{eM=M} \{\mathrm{Ann}_\Delta(\mathrm{Ext}^1_\Delta(M, \_))\}$, where the intersection

Received by the editors June 28, 1972.

AMS (MOS) subject classifications (1970). Primary 16A16, 16A18, 16A36; Secondary 13B20, 16A26, 16A32, 16A62, 16A64, 18G15, 18H15, 20C10.

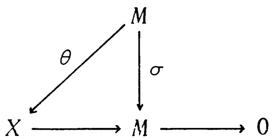
(1) The author wishes to express his appreciation to the University of Illinois for their support during his leave of absence from the University of Utah.

is over all left Λ-lattices M such that $eM = M$. Set $J_\Delta(\Lambda) = J_\Delta(\Lambda, 1)$, and $J_R(\Lambda, e) = J_\Delta(\Lambda, e) \cap R$.

Ideals related to $J_\Delta(\Lambda, e)$ have been studied by Jacobinski [5], Roggenkamp [6] and others.

Clearly $J_\Delta(\Lambda) = \Delta$ if and only if each left Λ-lattice is projective, that is, if and only if Λ is left hereditary. Let $\Lambda^e = \Lambda \otimes_R \Lambda^0$ be the enveloping algebra of Λ, so that the multiplication map $\epsilon: \Lambda^e \to \Lambda$, given by $\epsilon(a \otimes b) = ab$, is a left Λ^e-homomorphism; then ϵ induces the map $\epsilon_*: \mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda^e) \to \Delta$ by $\epsilon_*(f) = \epsilon f(1)$. The image $H_\Delta(\Lambda)$ of ϵ_* is an ideal of Δ, called the homological different of Λ (see Auslander and Goldman [1]). Since A is a separable K-algebra, it can be shown that $H_\Delta(\Lambda)$ is a nonzero ideal of Δ contained in $J_\Delta(\Lambda)$, and which spans the center $Z(A)$ of A over K (see for example [7, Chapter V]). It follows that $\mathrm{Ext}^1_\Lambda(M, N)$ is a torsion R-module for all left Λ-lattices M and all left Λ-modules N.

Let M be a left Λ-module. An endomorphism $\sigma \in \mathrm{End}_\Lambda(M)$ is called *projective* if, for every exact sequence $X \to M \to 0$ of left Λ-modules, there exists $\theta \in \mathrm{Hom}_\Lambda(M, X)$ such that the diagram



commutes.

Lemma 1.1. *Let M be a left Λ-module. Then $\sigma \in \mathrm{End}_\Lambda(M)$ is projective if and only if $\sigma\, \mathrm{Ext}^1_\Lambda(M, \_) = 0$.*

Proof. See Roggenkamp [6, Lemma 1].

If M is a left Λ-module and if $z \in \Delta$, then it follows from this lemma that $z\, \mathrm{Ext}^1_\Lambda(M, \_) = 0$ if and only if the left multiplication endomorphism λ_z of M is projective.

Our first result shows that $J_\Delta(\Lambda, e)$ can be computed directly in terms of $J_\Delta(\Lambda)$.

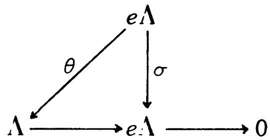
Theorem 1.2. *Let e be a central idempotent of A. Then $J_\Delta(\Lambda, e) = \{z \in \Delta: ze \in J_\Delta(\Lambda)\}$.*

Proof. To show the inclusion $\supseteq$, assume $z \in \Delta$ and $ze \in J_\Delta(\Lambda)$. Let M be a left Λ-lattice such that $eM = M$, so that the left multiplications λ_z and λ_{ze} on M are identical. Now $ze \in J_\Delta(\Lambda)$ implies that $\lambda_z = \lambda_{ze}$ is projective, hence $z \in J_\Delta(\Lambda, e)$.

For the inclusion $\subseteq$, fix $z \in J_\Delta(\Lambda, e)$. We will show first that $ze \in \Delta$.

Lemma 1.3. *Let e be a central idempotent of A. If $\sigma \in \mathrm{End}_\Lambda(e\Lambda)$ is projective, then $\sigma(e) \in \Lambda$.*

Proof. Let the epimorphism $\Lambda \to e\Lambda$ be given by $a \to ea$. Since σ is projective, there exists $\theta \in \mathrm{Hom}_\Lambda(e\Lambda, \Lambda)$ such that the diagram



commutes. In particular, $e\theta(e) = \sigma(e)$. Choose $0 \neq r \in R$ such that $re \in \Lambda$. Then $r\sigma(e) = re\theta(e) = \theta(re) = r\theta(e)$, and since Λ is R-torsion free, $\sigma(e) = \theta(e) \in \Lambda$, as desired.

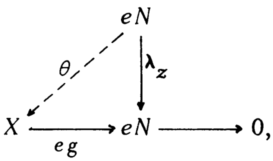
As a trivial consequence, we deduce the following well-known result about hereditary orders.

Corollary 1.4. *If Λ is hereditary, then Λ contains each central idempotent of A; in particular, Λ is the ring direct product of R-orders in simple K-algebras.*

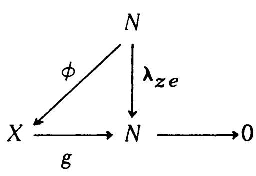
We continue with the proof of Theorem 1.2. Since $z \in J_\Delta(\Lambda, e)$, the left multiplication λ_z on $e\Lambda$ is projective. By the lemma, $ze = \lambda_z(e) \in \Lambda$, so in fact $ze \in \Delta$.

Now let N be a left Λ-lattice, and let $X \xrightarrow{g} N \to 0$ be an exact sequence of Λ-modules. We need to show that the left multiplication λ_{ze} is a projective endomorphism of N. The obvious map $N \to eN$ is an epimorphism, so we have the solid part of the diagram

(*)



where eg is the composite $X \xrightarrow{g} N \to eN$. Clearly $\lambda_z = \lambda_{ze}$ on eN, and $e(eN) = eN$. Since $z \in J_\Delta(\Lambda, e)$, we see that λ_z is a projective endomorphism of eN, so there exists $\theta \in \mathrm{Hom}\,(eN, X)$ such that the diagram (*) commutes. Now let $\phi = \theta e$ be the composite $N \to eN \xrightarrow{\theta} X$, so that $\phi(n) = \theta(en)$ for all $n \in N$. It is easy to check that the diagram



commutes, using the technique in the proof of Lemma 1.3. It follows that $ze \in J_\Delta(\Lambda)$, which completes the proof.

Corollary 1.5. *Let* e *be a central idempotent of* A. *Then* $eJ_\Delta(\Lambda) = eJ_\Delta(\Lambda, e) \subseteq J_\Delta(\Lambda)$.

Proof. By Theorem 1.2, $eJ_\Delta(\Lambda, e) \subseteq J_\Delta(\Lambda)$, so $eJ_\Delta(\Lambda, e) \subseteq eJ_\Delta(\Lambda)$. Clearly $J_\Delta(\Lambda) \subseteq J_\Delta(\Lambda, e)$. Therefore $eJ_\Delta(\Lambda) = eJ_\Delta(\Lambda, e) \subseteq J_\Delta(\Lambda)$, as desired.

Let e_1, e_2 be orthogonal central idempotents of A, and set $e = e_1 + e_2$. Then e is a central idempotent of A, and it is clear from the definition that $J_\Delta(\Lambda, e_i) \supseteq J_\Delta(\Lambda, e)$ for $i = 1, 2$. If $z \in J_\Delta(\Lambda, e_1) \cap J_\Delta(\Lambda, e_2)$, then by Theorem 1.2, $ze_1, ze_2 \in J_\Delta(\Lambda)$, and so $ze = ze_1 + ze_2 \in J_\Delta(\Lambda)$; therefore $z \in J_\Delta(\Lambda, e)$. We have thus proved the following intersection property.

Theorem 1.6. *Let* e_1, e_2 *be orthogonal central idempotents of* A. *Then*

$$J_\Delta(\Lambda, e_1) \cap J_\Delta(\Lambda, e_2) = J_\Delta(\Lambda, e_1 + e_2).$$

A *block* idempotent of a ring B is a nonzero central idempotent which cannot be expressed as the sum of two nonzero orthogonal central idempotents of B. Since the K-algebra A is semisimple, its block idempotents are in one-to-one correspondence with its simple ring direct factors, and every central idempotent of A is the sum of block idempotents. It follows from the above theorem by induction that if $e_1, \cdots, e_r$ are the block idempotents of A, then $J_\Delta(\Lambda, e) = \bigcap_{e_i e = e_i} \{J_\Delta(\Lambda, e_i)\}$ for any central idempotent e of A.

2. $J_\Delta(\Lambda)$ **for symmetric algebras.** We adhere to the notation and assumptions of §1. In addition, we shall use the following notation through the remainder of the paper:

Let $e_1, \cdots, r_r$ be the block idempotents of A. Set
$A_j = e_j A$,
$K_j = Z(A_j)$, the center of A_j,
$n_j = (A_j : K_j)^{1/2}$,
$R_j =$ the integral closure of R in K_j,
$U_j =$ the reduced trace of A_j over K_j,
$T_j =$ the trace of K_j over K,
$S_j = T_j \circ U_j$,
for all j such that $1 \leq j \leq r$.

Let B be a finitely generated S-algebra, where S is a commutative ring. We say the pair (B, ϕ) is a *symmetric* S-algebra if $\phi\colon B \longrightarrow \mathrm{Hom}_S(B, S)$ is an isomorphism as left B^e-modules, where $B^e = B \otimes_S B^0$. We say B is a symmetric algebra if there exists a ϕ such that (B, ϕ) is a symmetric algebra. It is well known that every finitely generated semisimple algebra over a field E is a symmetric E-algebra. Furthermore, if G is a finite group, then the group algebra SG is a symmetric S-algebra for any commutative ring S.

Let Γ be an R-order (in A) containing Λ, and set $(\Lambda:\Gamma)_\Lambda = \{a \in \Lambda: \Gamma a \subset \Lambda\}$. Then $(\Lambda:\Gamma)_\Lambda$ is an ideal in Λ, and it is also a left ideal in Γ. Set $(\Lambda:\Gamma)_\Delta = (\Lambda:\Gamma)_\Lambda \cap \Delta$.

Proposition 2.1 (Roggenkamp [6]). *If* Γ *is a hereditary R-order containing* Λ, *then* $(\Lambda:\Gamma)_\Delta \subseteq J_\Delta(\Lambda) \subseteq (\Lambda:\Gamma)_\Lambda \Gamma \cap \Delta$. *In particular if* $(\Lambda:\Gamma)_\Lambda\Gamma = (\Lambda:\Gamma)_\Lambda$, *then* $J_\Delta(\Lambda) = (\Lambda:\Gamma)_\Delta$.

We proceed to compute $(\Lambda:\Gamma)_\Lambda$ in case Λ is a symmetric R-algebra and Γ is a maximal R-order containing Λ, following Jacobinski [5].

Let (Λ, ϕ) be a symmetric R-algebra. We can extend ϕ to a left A^e-isomorphism $\phi: A \to \mathrm{Hom}_K(A, K)$. Since K has characteristic zero, the reduced trace S_j of A_j over K is nonzero, and it follows from [4, Theorem 1.3] that $\phi^{-1}(S_j)$ is a nonzero element in K_j for each j, $1 \le j \le r$. Let ρ be the character of the left regular module ${}_AA$. One checks that $\rho = \Sigma_j n_j S_j$, and it follows that $\Sigma_j n_j \phi^{-1}(S_j) = \phi^{-1}(\rho)$ is a unit in the center of A. (In [4] we showed that if (a_i), (b_i) are dual bases for A with respect to ϕ, then $\Sigma_i a_i b_i = \phi^{-1}(\rho)$.) Since Λ is integral over R, the restriction of ρ to Λ maps Λ into R, and so $\phi^{-1}(\rho)$ belongs to the center of Λ.

Definition. Let (Λ, ϕ) be a symmetric R-algebra, and let ρ be the character of the left regular module ${}_AA$. The *order* of (Λ, ϕ) is defined to be $\phi^{-1}(\rho)$.

Remarks. (1) If $G = \{g_1, g_2, \cdots, g_n\}$ is a finite group, then RG is a symmetric algebra with respect to an isomorphism ϕ such that (g_i), (g_i^{-1}) are dual bases with respect to ϕ. In this case the order of (RG, ϕ) is $\Sigma_i g_i g_i^{-1} = n$, the order of G. It is this fact that motivates the general definition.

(2) If Λ is a symmetric algebra with two Λ^e-isomorphisms $\phi_1, \phi_2: \Lambda \to \Lambda^*$, one can show that $\phi_1^{-1}(\rho)$ is a multiple of $\phi_2^{-1}(\rho)$ by a unit in the center Δ of Λ. Therefore the ideals of Δ generated by the orders of (Λ, ϕ_1) and (Λ, ϕ_2) are equal.

If L is a finitely generated R-submodule of A such that $KL = A$, set $L^\phi = \{a \in A: \phi(1)(aL) \subseteq R\}$. Then $L \to L^\phi$ reverses strict inclusions, and $(L^\phi)^\phi = L$. Since $\phi: \Lambda \to \Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ is an isomorphism, one checks that $\Lambda^\phi = \Lambda$. Now if Γ is any order containing Λ, then $\Lambda^\phi\Gamma = \Gamma$ is the smallest right Γ-lattice containing Λ^ϕ, so Γ^ϕ is the largest left Γ-lattice contained in Λ; that is, $\Gamma^\phi = (\Lambda:\Gamma)_\Lambda$. Since $\phi(1)(xa) = \phi(1)(ax)$ for all $x, a \in A$ (see [4, Theorem 1.3]), it follows that Γ^ϕ is a two-sided ideal of Γ contained in Λ.

Now assume that Γ is a maximal R-order containing Λ, and as before let c denote the order of (Λ, ϕ). Since Γ is maximal, we have a decomposition $\Gamma = \Gamma_1 + \cdots + \Gamma_r$, where each Γ_j is a maximal order in A_j. Note also that R_j is the center of Γ_j for each j. Now $\rho = \Sigma_j n_j S_j$, so it follows easily that

$\{a \in A: \rho(a\Gamma) \subseteq R\} = \Sigma_j (1/n_j) d_{S_j}^{-1}(\Gamma_j)$, where we use the notation of [7, Chapter V, §1] by writing $d_{S_j}^{-1}(\Gamma_j) = \{a \in A_j: S_j(a\Gamma_j) \subseteq R\}$. Similarly define $d_{U_j}^{-1}(\Gamma_j)$ and $d_{T_j}^{-1}(R_j)$.

Now $\phi(1) = c^{-1}\phi(c) = c^{-1}\rho$, so $(\Lambda:\Gamma)_\Lambda = \Gamma^\phi = \{a \in A: \phi(1)(a\Gamma) \subseteq R\} = \{a \in A: (c^{-1}\rho)(a\Gamma) \subseteq R\} = c\{a \in A: \rho(a\Gamma) \subseteq R\} = \Sigma_j (c/n_j) d_{S_j}^{-1}(\Gamma_j)$, by the above paragraph. From the equalities $d_{S_j}^{-1}(\Gamma_j) = d_{U_j}^{-1}(\Gamma_j) d_{T_j}^{-1}(R_j)$, we have the following:

Lemma 2.2. *Let Γ be a maximal order containing Λ. If (Λ, ϕ) is a symmetric R-algebra of order c, then*

$$(\Lambda:\Gamma)_\Lambda = \sum_j (c/n_j)\, d_{U_j}^{-1}(\Gamma_j)\, d_{T_j}^{-1}(R_j),$$

and

$$(\Lambda:\Gamma)_\Lambda \Gamma = (\Lambda:\Gamma)_\Lambda.$$

Since K is an algebraic number field, $d_{U_j}^{-1}(\Gamma_j) \cap K_j = R_j$ for each j (see [7, p. 270]). From this, together with our observation that $\Gamma^\phi = (\Lambda:\Gamma)_\Lambda$ is a two-sided ideal of Γ, we apply Proposition 2.1 to obtain a characterization of $J_\Delta(\Lambda)$ for symmetric algebras (compare Jacobinski [5] and Roggenkamp [6]).

Theorem 2.3. *Let (Λ, ϕ) be a symmetric R-algebra of order c. Then $J_\Delta(\Lambda) = \Sigma_j (c/n_j) d_{T_j}^{-1}(R_j)$. Furthermore, if e is a central idempotent of A, then*

$$eJ_\Delta(\Lambda) = \sum_{e_j e = e_j} (c/n_j) d_{T_j}^{-1}(R_j),$$

and

$$J_R(\Lambda, e) = \bigcap_{e_j e = e_j} \{(c/n_j) d_{T_j}^{-1}(R_j) \cap K\}.$$

This theorem has an obvious interpretation for group algebras. In particular, assume R is a discrete valuation ring (DVR) with maximal ideal πR, and let G be a finite group of order n. Then n is also the order of the symmetric algebra RG. Assume that K is a splitting field for $KG = A$. Let e be a block idempotent of RG, and write $e = e_1 + \cdots + e_s$ as a sum of block idempotents of A. For each j, let d_j be the nonnegative integer such that $\pi^{d_j} R = (n/n_j)R$, where n_j^2 is the degree of $e_j A$ over its center K. Set $d = \max\{d_j: 1 \le j \le s\}$. Since K is a splitting field for A, each $R_j = R$ (in the notation of this section), so by the above theorem, $J_R(RG, e) = \bigcap_j \pi^{d_j} R = \pi^d R$. Note that d is the defect of the block idempotent e of RG as defined in Curtis and Reiner [2, §86].

3. A generalization of the homological different. In this section we define certain ideals of Δ which are related to the homological different, and which bear close resemblance to the ideals $J_\Delta(\Lambda, e)$ studied in the previous sections. Notation and assumptions of the previous sections will be retained. Recall that Λ is defined to be a separable R-algebra if Λ is a projective Λ^e-module. The reader is directed to Auslander and Goldman [1] for pertinent facts about separability.

The multiplication map $\epsilon: \Lambda^e \to \Lambda$ induces the map $\epsilon_*: \mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda^e) \to \Delta$ by $\epsilon_*(f) = \epsilon f(1)$, and the image $H_\Delta(\Lambda)$ of ϵ_* is called the *homological different* of Λ. It is easy to show that $H_\Delta(\Lambda) = \mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \_))$, and $H_\Delta(\Lambda) \subseteq J_\Delta(\Lambda)$.

Definition. Let e be a central idempotent of A. Define $M_\Delta(\Lambda, e) = \bigcap_{\Lambda\subseteq\Gamma}\{\mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(e\Gamma, \_))\}$, where the intersection is taken over all R-orders Γ containing Λ. Set $M_\Delta(\Lambda) = M_\Delta(\Lambda, 1)$, and $M_R(\Lambda, e) = M_\Delta(\Lambda, e) \cap R$.

It is clear from the definition that $M_\Delta(\Lambda) \subseteq H_\Delta(\Lambda)$. Since separable orders are maximal [1, Proposition 7.1], it follows that $M_\Delta(\Lambda) = \Delta$ if and only if Λ is separable over R.

We have an obvious analogue to Theorem 1.2.

Theorem 3.1. *Let e be a central idempotent of A. Then* $M_\Delta(\Lambda, e) = \{z \in \Delta: ze \in M_\Delta(\Lambda)\}$.

Proof. To show the inclusion $\supseteq$, assume $z \in \Delta$ and $ze \in M_\Delta(\Lambda)$. Let Γ be an R-order containing Λ, and set $\Gamma_0 = e\Gamma + (1-e)\Gamma$. Then Γ_0 is an R-order containing Λ such that $e\Gamma_0 = e\Gamma$. Since $\mathrm{Ext}^1_{\Lambda^e}(\Gamma_0, \_) \cong \mathrm{Ext}^1_{\Lambda^e}(e\Gamma, \_) \oplus \mathrm{Ext}^1_{\Lambda^e}((1-e)\Gamma, \_)$, we see that $\mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(e\Gamma, \_)) \supseteq \mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(\Gamma_0, \_)) \supseteq M_\Delta(\Lambda)$, hence $ze \in \mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(e\Gamma, \_))$. Now the left multiplications λ_z and λ_{ze} on $e\Gamma$ are identical, so $z \in \mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(e\Gamma, \_))$. Thus $z \in M_\Delta(\Lambda, e)$.

For the inclusion $\subseteq$, fix $z \in M_\Delta(\Lambda, e)$. Then the left multiplication λ_z on $e\Lambda$ is projective, so essentially from Lemma 1.3 we see that $ze = \lambda_z(e) \in \Delta$. The remainder of the proof now parallels that of Theorem 1.2.

Corollary 3.2. *Let e be a central idempotent of A. Then* $eM_\Delta(\Lambda) = eM_\Delta(\Lambda, e) \subseteq M_\Delta(\Lambda)$.

Proof. Copy the proof of Corollary 1.5.

We proceed to give an explicit computation of $M_\Delta(\Lambda)$ in case Λ is a symmetric algebra.

Now R is a Dedekind domain, so Λ is finitely generated and projective over R, and the map $\gamma: \Lambda^e \to \mathrm{Hom}_R(\Lambda^*, \Lambda)$ given by $\gamma(a \otimes b)(f) = af(b)$, for $a \otimes b \in \Lambda^e$, and for all $f \in \Lambda^* = \mathrm{Hom}_R(\Lambda, R)$, is a (Λ^e, Λ^e)-bimodule isomorphism. Here $\mathrm{Hom}_R(\Lambda^*, \Lambda)$ is a (Λ^e, Λ^e)-bimodule by defining $[(a \otimes b)\theta](f) = a\theta(bf)$ and $[\theta(a \otimes b)](f) = \theta(fb)a$, where $a \otimes b \in \Lambda^e$, $\theta \in \mathrm{Hom}_R(\Lambda^*, \Lambda)$, and $f \in \Lambda^*$.

Let Γ be an R-order containing Λ, and consider the obvious Λ^e-homomorphism $\Lambda \to \Gamma$ given by inclusion. We obtain the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e) \otimes_{\Lambda^e} \Gamma & \xrightarrow{\mu^\Gamma} & \mathrm{Hom}_{\Lambda^e}(\Gamma, \Gamma) \\
\uparrow \sigma^\Gamma & & \uparrow \tau^\Gamma \\
\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e) & \xrightarrow{\epsilon_*^\Gamma} & \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda),
\end{array}$$

where $\sigma^\Gamma(f) = f \otimes 1$, τ^Γ is induced from $\Lambda \to \Gamma$, ϵ_*^Γ is induced from the multiplication ϵ: $\Lambda^e \to \Lambda$, and $[\mu^\Gamma(f \otimes x)](y) = f(y)x$. Note that τ^Γ is one-to-one.

Lemma 3.3 (Roggenkamp [6]). *The image of μ^Γ is the set of all projective Λ^e-endomorphisms of Γ.*

We may regard $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$ as a right Λ^e-submodule of the right A^e-module $\mathrm{Hom}_{A^e}(A, A^e)$. Since $\Gamma \to A^e$ given by $x \to x \otimes 1$ is a ring homomorphism, $\mathrm{Hom}_{A^e}(A, A^e)$ becomes a right Γ-module.

Lemma 3.4. *Let Γ be an R-order containing Λ. If $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)\Gamma \subseteq \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$, then the restriction τ_0^Γ: $\mathrm{Im}(\epsilon_*^\Gamma) \to \mathrm{Im}(\mu^\Gamma)$ of τ^Γ to $\mathrm{Im}(\epsilon_*^\Gamma)$ is an isomorphism.*

Proof. It is clear that τ^Γ, and therefore τ_0^Γ, is one-to-one. The commutativity of the above diagram shows that τ_0^Γ maps into $\mathrm{Im}(\mu^\Gamma)$; thus we only need to show that it is onto. Let $f \in \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$ and $x \in \Gamma$. Choose $0 \neq r \in R$ such that $rx \in \Lambda$. Then $r(f \otimes x) = f \otimes rx = f \otimes (rx \otimes 1)1 = f(rx \otimes 1) \otimes 1 = frx \otimes 1$ in $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e) \otimes_{\Lambda^e} \Gamma$. Since $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)\Gamma \subseteq \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$, we have that $fx \in \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$, and so from above $r(f \otimes x) = r(fx \otimes 1)$ in $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e) \otimes_{\Lambda^e} \Gamma$. Therefore $r\mu^\Gamma(f \otimes x) = r\mu^\Gamma(fx \otimes 1)$, and since $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Gamma)$ is R-torsion free, $\mu^\Gamma(f \otimes x) = \mu^\Gamma(fx \otimes 1) = \mu^\Gamma \sigma^\Gamma(fx) = \tau_0^\Gamma \epsilon_*^\Gamma(fx)$. It follows that τ_0^Γ is onto, as desired.

Remarks. (1) The hypotheses of the above lemma are clearly satisfied if $\Lambda = \Gamma$.

(2) The function $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda) \to (\Lambda : \Gamma)_\Lambda$ given by $g \to g(1)$ is an isomorphism, which we will regard as an identification; thus $\mathrm{Im}(\epsilon_*^\Gamma) \subseteq (\Lambda : \Gamma)_\Lambda$.

Lemma 3.5. *Let Γ be an R-order containing Λ. Assume (Λ, ϕ) is a symmetric R-algebra, and let (a_i), (b_i) be dual bases of A with respect to ϕ. Then θ^Γ: $(\Lambda : \Gamma)_\Lambda \to \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$ defined by $\theta^\Gamma(a)(x) = \Sigma_i\, xb_i a \otimes a_i$ is a right Λ^e-isomorphism. In particular, $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)\Gamma \subseteq \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$.*

Proof. Define ψ: $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e) \to (\Lambda : \Gamma)_\Lambda$ by $\psi(f) = [\gamma(f(1))](\phi(1))$, where γ: $\Lambda^e \to \mathrm{Hom}_R(\Lambda^*, \Lambda)$ is the isomorphism defined after Corollary 3.2. It is relatively routine to check that ψ is the inverse of θ^Γ, which shows that θ^Γ is an isomorphism. Now by Lemma 2.2, $(\Lambda : \Gamma)_\Lambda \Gamma \subseteq (\Lambda : \Gamma)_\Lambda$, so it follows that $\mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)\Gamma \subseteq \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$, as desired.

Let us continue to assume the hypotheses of Lemma 3.5. Now $\epsilon_*^\Gamma\colon \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e) \to (\Lambda:\Gamma)_\Delta$ is given by $\epsilon_*^\Gamma(g) = \epsilon\, g(1)$. By the lemma, $\theta^\Gamma\colon (\Lambda:\Gamma)_\Lambda \to \mathrm{Hom}_{\Lambda^e}(\Gamma, \Lambda^e)$ is given by $\theta^\Gamma(a)(x) = \Sigma_i\, x b_i a \otimes a_i$, so for any $a \in (\Lambda:\Gamma)_\Lambda$, $\epsilon_*^\Gamma(\theta^\Gamma(a)) = \epsilon[\theta^\Gamma(a)(1)] = \Sigma_i\, b_i a a_i$. (Observe that $\epsilon_*^\Gamma\theta^\Gamma$ coincides with the Gaschütz-Ikeda operator [2, §71].) Since θ^Γ is an isomorphism, $\mathrm{Im}(\epsilon_*^\Gamma\theta^\Gamma) = \mathrm{Im}(\epsilon_*^\Gamma) \cong \mathrm{Im}(\mu^\Gamma)$, by Lemmas 3.5 and 3.4. It is now easy to see that, for any $z \in \Delta$, the left multiplication λ_z on Γ is a projective Λ^e-endomorphism if and only if $z \in \mathrm{Im}(\epsilon_*^\Gamma) = \mathrm{Im}(\epsilon_*^\Gamma\theta^\Gamma)$, that is, if and only if $z = \Sigma_i\, b_i a a_i$ for some $a \in (\Lambda:\Gamma)_\Lambda$. We therefore have

Lemma 3.6. *Let Γ be an R-order containing Λ, and let (Λ, ϕ) be a symmetric R-algebra with dual bases (a_i), (b_i) for A with respect to ϕ. Then*

$$\mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(\Gamma, -)) = \left\{\sum_i b_i a a_i : a \in (\Lambda:\Gamma)_\Lambda\right\}.$$

Corollary 3.7. *If (Λ, ϕ) is a symmetric R-algebra, then $M_\Delta(\Lambda) = \bigcap_{\Lambda\subseteq\Gamma}\{\mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(\Gamma, \_))\}$, where the intersection is over all maximal R-orders containing Λ.*

Proof. If Γ, Γ' are R-orders such that $\Lambda \subseteq \Gamma \subseteq \Gamma'$, then plainly $(\Lambda:\Gamma')_\Lambda \subseteq (\Lambda:\Gamma)_\Lambda$. By the above lemma, $\mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(\Gamma', \_)) \subseteq \mathrm{Ann}_\Delta(\mathrm{Ext}^1_{\Lambda^e}(\Gamma, \_))$. The corollary now follows directly from the definition of $M_\Delta(\Lambda)$.

Lemma 3.8. *Let (Λ, ϕ) be a symmetric algebra of order c, and let (a_i), (b_i) be dual bases for A with respect to ϕ. Then for any $a \in A$, $\Sigma_i\, b_i a a_i = \Sigma_j\,(c/n_j)U_j(a)$.*

Proof. Fix $a \in A$, and write $\Sigma_i\, b_i a a_i = \Sigma_k\, \alpha_k$, where each $\alpha_k \in K_k$, the center of A_k. By the discussion following Proposition 2.1, we have that $c = \Sigma_i\, a_i b_i$, and c belongs to Δ. Therefore for any j, $cU_j(a) = U_j(ca) = U_j(\Sigma_i\, a_i b_i a) = U_j(\Sigma_i\, b_i a a_i) = U_j(\Sigma_k\, \alpha_k) = U_j(\alpha_j) = \alpha_j n_j$. By dividing by n_j, this implies that $\alpha_j = (c/n_j)U_j(a)$, as desired.

Now let Γ be a maximal R-order containing Λ, and assume that (Λ, ϕ) is a symmetric R-algebra of order c, with dual bases (a_i), (b_i) of A with respect to ϕ. By Lemma 2.2, $(\Lambda:\Gamma)_\Lambda = \Sigma_j\,(c/n_j)d_{U_j}^{-1}(\Gamma_j)d_{T_j}^{-1}(R_j)$. Recall that $\epsilon_*^\Gamma\theta^\Gamma\colon a \to \Sigma_i b_i a a_i = \Sigma_j(c/n_j)U_j(a)$ as above. It follows that the image of $(\Lambda:\Gamma)_\Lambda$ under $\epsilon_*^\Gamma\theta^\Gamma$ is

$$\sum_j (c/n_j)^2 d_{T_j}^{-1}(R_j)U_j(d_{U_j}^{-1}(\Gamma_j)).$$

Since R is a Dedekind domain, it is easy to see that $U_j(d_{U_j}^{-1}(\Gamma_j)) = R_j$. Therefore

$$\mathrm{Im}(\epsilon_*^\Gamma\theta^\Gamma) = \sum_j (c/n_j)^2 d_{T_j}^{-1}(R_j).$$

Observe that this expression is independent of the maximal order Γ. We can now apply Lemma 3.6 and Corollary 3.7 to obtain the following characterization of $M_\Delta(\Lambda)$.

Theorem 3.9. *Let (Λ, ϕ) be a symmetric R-algebra of order c. Then $M_\Delta(\Lambda) = \Sigma_j (c/n_j)^2 d_{T_j}^{-1}(R_j)$. Furthermore, if e is a central idempotent in A, then*

$$eM_\Delta(\Lambda) = \sum_{e_je=e_j} (c/n_j)^2 d_{T_j}^{-1}(R_j),$$

and

$$M_R(\Lambda, e) = \bigcap_{e_je=e_j} \{(c/n_j)^2 d_{T_j}^{-1}(R_j) \cap K\}.$$

We can use this characterization of $M_\Delta(\Lambda)$ to give the following theorem about separability:

Theorem 3.10. *Let e be a central idempotent of A. Then the following statements are equivalent:*

(a) *$e \in \Lambda$ and $e\Lambda$ is separable over R,*

(b) *$e\Lambda$ is a projective Λ^e-module,*

(c) *$M_\Delta(\Lambda, e) = \Delta$.*

If in addition, (Λ, ϕ) is a symmetric algebra of order c, then these are equivalent to the following statement:

(d) *(c/n_j) is a unit in R_j for each j such that $e_je = e_j$.*

Proof. The proof of the equivalence of (a), (b) and (c) is left to the reader. Now assume (a), (b) and (c). Then $e\Lambda$ is a maximal R-order by [1, Proposition 7.1], and so R_j is the center of $e_j\Lambda$ for each j such that $e_je = e_j$. Since each $e_j\Lambda$ is R-separable, it follows that R_j is R-separable [1, Theorem 2.3], and one can check in this case that $d_{T_j}^{-1}(R_j) = R_j$. By Theorem 3.9, $e\Delta = eM_\Delta(\Lambda, e) = \Sigma_{e_je=e_j} (c/n_j)^2 R_j$, and it follows that (c/n_j) is a unit in R_j for each j such that $e_je = e_j$. This proves (d). Now assume (d). Fix a j such that $e_je = e_j$, so that (c/n_j) is a unit in R_j; since $(c/n_j)^2 d_{T_j}^{-1}(R_j)$ is an ideal in the domain R_j and (c/n_j) is a unit, it follows that $(c/n_j)R_j = d_{T_j}^{-1}(R_j) = R_j$. Therefore $M_R(\Lambda, e) = R$, and so $1 \in M_R(\Lambda, e)$. Thus $1 \in M_\Delta(\Lambda, e)$, and so $M_\Delta(\Lambda, e) = \Delta$, which shows that (d) implies (c). This completes the proof of the theorem.

We conclude this section with an explicit computation of $M_R(\Lambda)$ when Λ is the group algebra RG of a finite group G.

Proposition 3.11. *Let G be a finite group of order n. Then $M_R(RG) = n^2R$.*

Proof. We have already observed that RG is a symmetric algebra of order n. It is clear from Theorem 3.9, that $M_R(RG) \supseteq n^2R$. If e_1 is the block idempotent of

KG corresponding to the KG-module K with trivial G-action, then $n_1 = 1$ and $R_1 = R$. Therefore $(n/n_1)^2 d_{T_1}^{-1}(R_1) \cap K = n^2 R$. Theorem 3.9 now implies that $M_R(RG) \subseteq n^2 R$, and the corollary follows.

4. **A characterization of separable orders.** Assume, as in the previous sections, that R is a Dedekind domain with quotient field K, and that Λ is an R-order in the separable K-algebra A. If p is a maximal ideal of R, let R_p denote the localization of R at p. Similarly set $\Lambda_p = R_p \otimes_R \Lambda$. The main theorem of this section is the following characterization of separable orders.

Theorem 4.1. *The R-order Λ is separable over R if and only if the following statements hold*:

(1) *the center $Z(\Lambda)$ of Λ is separable over R,*

(2) *Λ is a symmetric R-algebra,*

(3) *the natural map $Z(\Lambda_p) \to Z(\Lambda_p/p\Lambda_p)$ is onto for each maximal ideal p of R.*

Proof. Assume first that Λ is separable over R. We have already observed that $Z(\Lambda)$ is separable over R, so (1) is established. Endo and Watanabe [3] have shown that Λ is a symmetric algebra, which proves (2). Now (3) follows immediately from the following.

Proposition 4.2. *For any R-order Λ in A, $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda) = 0$ if and only if the natural map $Z(\Lambda_p) \to Z(\Lambda_p/p\Lambda_p)$ is onto for each maximal ideal p of R.*

Proof. Consider first the case where R is a DVR with maximal ideal πR, and set $\bar{\Lambda} = \Lambda/\pi\Lambda$. By applying the functor $\mathrm{Hom}_{\Lambda^e}(\Lambda, \_)$ to the exact sequence $0 \to \Lambda \xrightarrow{\pi} \Lambda \to \bar{\Lambda} \to 0$ of left Λ^e-modules, we see that $Z(\Lambda) \to Z(\bar{\Lambda}) \to \mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda) \xrightarrow{\pi} \mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda)$ is exact, where π is used here to denote multiplication. Now if $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda) = 0$, it is clear that $Z(\Lambda) \to Z(\bar{\Lambda})$ is onto. Conversely, if $Z(\Lambda) \to Z(\bar{\Lambda})$ is onto, then $0 \to \mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda) \xrightarrow{\pi} \mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda)$ is exact; but this surely implies that $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda) = 0$, since otherwise multiplication by π on the torsion R-module $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda)$ would not be one-to-one.

For the general case, we need only observe that $\mathrm{Ext}^1_{\Lambda^e}(\Lambda, \Lambda) \cong \bigoplus_p \mathrm{Ext}^1_{\Lambda_p^e}(\Lambda_p, \Lambda_p)$, where the sum is over all maximal ideals p of R. This concludes the proof of the proposition.

Returning now to the proof of Theorem 4.1, assume that conditions (1), (2), and (3) hold. We must show that Λ is separable over R. By [1, Corollary 4.5], it is sufficient to show that Λ_p is separable over R_p for each maximal ideal p of R. We leave it to the reader to verify that conditions (1) and (2) imply their local versions. We may therefore assume that R is a DVR with maximal ideal

πR, $Z(\Lambda)$ is separable over R, Λ is a symmetric R-algebra, and the natural map $Z(\Lambda) \to Z(\Lambda/\pi\Lambda)$ is onto. It is easy to see that $\overline{\Lambda}$ is a symmetric $\overline{R}$-algebra, where $\overline{R} = R/\pi R$ and $\overline{\Lambda} = \Lambda/\pi\Lambda$. Since $Z(\Lambda) \to Z(\overline{\Lambda})$ is onto, [1, Theorem 4.7] implies that $Z(\overline{\Lambda})$ is separable over the field $\overline{R}$. To show that Λ is separable over R, by [1, Theorem 4.7] it suffices to show that $\overline{\Lambda}$ is separable over $\overline{R}$. The proof is complete by establishing the following.

Theorem 4.3. *Let F be a field and let B be a symmetric F-algebra. Then B is semisimple if and only if its center $Z(B)$ is semisimple. Moreover, B is separable over F if and only if its center $Z(B)$ is separable over F.*

Proof. If B is semisimple, so is $Z(B)$. So assume conversely that $Z(B)$ is semisimple, and let J denote the radical of B. Then $J \cap Z(B) = 0$. By applying $\operatorname{Hom}_{B^e}(B, \_)$ to the exact sequence $0 \to J \to B \to B/J \to 0$ of left B^e-modules, we see that $0 \to J \cap Z(B) \to Z(B) \to Z(B/J)$ is exact. Since $J \cap Z(B) = 0$, the map $Z(B) \to Z(B/J)$ is one-to-one. Now dualize with respect to F by applying $\operatorname{Hom}_F(\_, F) = (\_)^*$ to obtain the exact sequence $0 \to (B/J)^* \to B^* \to J^* \to 0$ of left B^e-modules. Now B/J is semisimple, so B/J is a symmetric F-algebra. It follows that $(B/J)^* \cong B/J$ as left B^e-modules, and $B^* \cong B$ as left B^e-modules by assumption. Thus we have an exact sequence $0 \to B/J \to B$. Again applying $\operatorname{Hom}_{B^e}(B, \_)$ we have that $0 \to Z(B/J) \to Z(B)$ is exact. It follows by counting dimensions, using the previously established monomorphism $Z(B) \to Z(B/J)$, that $Z(B/J) \to Z(B)$ is onto. Hence 1 is in the image of $B/J \to B$. Since this image is an ideal of B, the map $B/J \to B$ is an epimorphism. This is impossible unless $J = 0$, by counting dimensions. Hence B is semisimple. The remainder of the theorem now follows from the characterization of a separable F-algebra as an F-algebra which is semisimple and whose center is separable over F.

Corollary 4.4. *Let A be a central simple K-algebra. Then Λ is separable over R if and only if the following statements hold:*

(1) *Λ is a symmetric R-algebra,*

(2) *the natural map $Z(\Lambda_p) \to Z(\Lambda_p/p\Lambda_p)$ is onto for each maximal ideal p of R.*

Corollary 4.5. *Let e be a central idempotent of Λ, and assume Λ is a symmetric R-algebra such that the natural map $Z(\Lambda_p) \to Z(\Lambda_p/p\Lambda_p)$ is onto for each maximal ideal p of R. If K is a splitting field for eA, then the following statements are equivalent:*

(a) *$e\Lambda$ is separable over R.*

(a′) $M_\Delta(\Lambda, e) = \Delta$.

(b) *$e\Lambda$ is hereditary.*

(b′) $J_\Delta(\Lambda, e) = \Delta$.

(c) *$e_j \in \Lambda$ for all block idempotents e_j of A such that $e_j e = e_j$.*

Proof. The equivalences (a) $\Leftrightarrow$ (a′) and (b) $\Leftrightarrow$ (b′) are routine. Clearly (a) implies (b), and (b) implies (c) by Corollary 1.4. We will prove that (c) implies (a). Let e_j be a block idempotent of A such that $e_j e = e_j$. Then $e_j \in \Lambda$, and $e_j A$ is K-central simple because K is a splitting field for eA. One checks that hypotheses (1) and (2) of Corollary 4.4 are satisfied by $e_j \Lambda$, since $e_j \in \Lambda$, and so $e_j \Lambda$ is separable over R. Since e is the sum of those block idempotents e_j of A such that $e_j e = e_j$, if follows that $e\Lambda$ is separable over R, completing the proof.

Observe that the above hypotheses are satisfied if Λ is the group algebra RG of a finite group G, and if K is a splitting field for KG. The corollary is false if K is not a splitting field: For example, $\mathbf{Z}[i]$ is a commutative, symmetric $\mathbf{Z}$-order in the simple $\mathbf{Q}$-algebra $\mathbf{Q}[i]$, where $i^2 = -1$, so (c) is clearly satisfied in the above corollary (with $e = 1$), but $\mathbf{Z}[i]$ is not separable over $\mathbf{Z}$.

Examples. (1) Let R be a DVR with maximal ideal πR and quotient field K, and let Λ be the set of all matrices of the form

$$\begin{pmatrix} a & \pi b \\ c & d \end{pmatrix}$$

for $a, b, c, d \in R$. Then Λ is a hereditary order in the central simple algebra $(K)_2$ of two-by-two matrices over K, but Λ is not maximal, hence it is not separable over R. One can check that $\overline{\Lambda} = \Lambda/\pi\Lambda$ is not a symmetric algebra (it is a Frobenius algebra), but $Z(\Lambda) \to Z(\overline{\Lambda})$ is onto. This shows that the hypothesis that Λ be symmetric cannot be deleted from Corollary 4.4.

(2) Now let $R = \mathbf{Z}_{(2)}$, the localization of the ring $\mathbf{Z}$ of integers at the maximal ideal (2), and let Λ be the R-algebra freely generated by $\{1, a, b, c\}$, subject to the following multiplication:

	1	a	b	c
1	1	a	b	c
a	a	-1	c	$-b$
b	b	$-c$	1	$-a$
c	c	b	a	1

One can check that Λ is a twisted group algebra over R of the Klein four-group G, with the obvious factor set. It follows that Λ is a symmetric algebra. Moreover, if K denotes the rational field, so that K is the quotient field of R, then $K \otimes_R \Lambda = A$ is a K-central simple algebra. Now the residue class algebra $\overline{\Lambda} = \Lambda/2\Lambda$ is the ordinary group algebra $\overline{R}G$ over $\overline{R} \cong \mathbf{Z}/(2)$, so $\overline{\Lambda}$ is not semisimple.

Hence Λ is not separable over R. It is obvious that Λ is noncommutative while $\overline{\Lambda}$ is commutative, so $Z(\Lambda) \to Z(\overline{\Lambda})$ is not onto. This shows that condition (2) of Corollary 4.4 cannot be deleted.

Using [3], the proof of Theorem 4.1 may be modified to apply in case Λ is a finitely generated projective faithful algebra over an arbitrary commutative ring R.

REFERENCES

1. M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. 97 (1960), 367–409. MR 22 #12130.

2. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. 11, Interscience, New York, 1962. MR 26 #2519.

3. S. Endo and Y. Watanabe, *On separable algebras over a commutative ring*, Osaka J. Math. 4 (1967), 233–242. MR 37 #2796.

4. T. V. Fossum, *Characters and centers of symmetric algebras*, J. Algebra 16 (1970), 4–13. MR 41 #6894.

5. H. Jacobinski, *On extensions of lattices*, Michigan Math. J. 13 (1966), 471–475. MR 34 #4377.

6. K. W. Roggenkamp, *Projective homomorphisms and extensions of lattices*, J. Reine Angew. Math. 246 (1971), 41–45. MR 43 #248.

7. K. W. Roggenkamp and V. Huber-Dyson, *Lattices over orders, I*, Lecture Notes in Math., vol. 115, Springer-Verlag, New York, 1970.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

Current address: Department of Mathematics, University of Utah, Salt Lake City, Utah 84112